

# COMMUTATOR SUBGROUPS OF WELDED BRAID GROUPS

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**ABSTRACT.** Let  $WB_n$  be the welded (or loop) braid group on  $n$  strands,  $n \geq 3$ . We investigate commutator subgroup of  $WB_n$ . We prove that the commutator subgroup  $WB'_n$  is finitely generated and Hopfian. We show that  $WB'_n$  is perfect if and only if  $n \geq 5$ . We also compute finite presentations for commutator subgroups of flat virtual and flat welded braid groups. Along the way, we investigate adorability of these generalised braid groups.

## 1. INTRODUCTION

A *loop braid group* or *welded braid group* or *permutation-braid group* is an extension of the classical braid group. These groups were independently introduced by several authors at various time, and each with different name. The name ‘loop braid’ is due to Baez, Wise and Crans [BWC07] and the name ‘welded braid’ is due to Fenn, Rimányi and Rourke [FRR97]. The welded braids are closely related to the virtual braids that was introduced by Kauffman [Kau99]. Just like closures of classical braids represent classical knots and links, the closure of welded braids represent the welded knots and links, see [Kam07]. This class of generalized braid groups can also be viewed as automorphisms of free groups, known as the *symmetric automorphisms* of a free group, cf. [Col89]. See the recent article [Dam16] for several formulations of the welded braid groups and their equivalence.

Kauffman introduced *flat* category of generalized braids those are certain quotients of the virtual braid group. The flat welded (resp. flat virtual) braid group on  $n$  strands is obtained by quotienting the welded (resp. virtual) braid group on  $n$  strands by the normal closure of the squares of the embedded classical braid generators. For example, see [KL04] for a detailed comparison of all these generalized braid groups and their presentations.

In this paper we aim to investigate the commutator subgroups of the welded braids and their flat quotients. The commutator subgroups of the classical braid groups are well studied. Gorin and Lin [GL69] obtained a finite presentation of the commutator subgroup  $B'_n$  of the classical braid groups  $B_n$ . Simpler presentation of  $B'_n$  was obtained by Savushkina [Sav93]. These results were generalized by Zinde [Zin75] to obtain presentations of several spherical type Coxeter groups. Mulholland and Rolfsen [MR] has also obtained similar presentations for these groups. Orevkov [Ore12] improved the results of Zinde. It is interesting to investigate commutator subgroups of other generalized braid groups, and we attempt so in this paper.

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Let  $WB_n$  denote the welded braid group of  $n$  strands. It is well-known that  $WB_n$  is generated by a set of  $2(n-1)$  generators:  $\{\sigma_i, \rho_i, i = 1, 2, \dots, n-1\}$  that satisfy the following set of relations:

(1) *The braid relations:*

$$\begin{aligned}\sigma_i \sigma_j &= \sigma_j \sigma_i \text{ if } |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}\end{aligned}$$

(2) *The symmetric relations:*

$$\begin{aligned}\rho_i^2 &= 1 \\ \rho_i \rho_j &= \rho_j \rho_i, \text{ if } |i - j| > 1 \\ \rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1}\end{aligned}$$

(3) *The mixed relations:*

$$\begin{aligned}\sigma_i \rho_j &= \rho_j \sigma_i, \text{ if } |i - j| > 1 \\ \rho_i \rho_{i+1} \sigma_i &= \sigma_{i+1} \rho_i \rho_{i+1}\end{aligned}$$

(4) *The forbidden relations:*

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}$$

If we add the *flat* relation:  $\sigma_i^2 = 1$ ,  $1 \leq i \leq n-1$ , in the above presentation, we get the *flat welded braid group*, denoted by  $FWB_n$ . The *flat virtual braid group*, denoted by  $FVB_n$ , is an extension of  $FWB_n$ , which is obtained by removing the forbidden relations from its presentation.

We investigate the commutator subgroup of  $WB_n$ . Recall that a group  $G$  is called *perfect* if it is equal to its commutator subgroup. We prove the following:

**Theorem 1.1.** *Let  $WB'_n$  denote the commutator subgroup of the welded braid group  $WB_n$ .*

- (i)  *$WB'_n$  is a finitely generated group for all  $n \geq 3$ . For  $n \geq 7$ , the rank of  $WB'_n$  is at most  $1 + 2(n-3)$ , and for  $3 \leq n \leq 6$ , the rank is at most  $4 + 2(n-3)$ .*
- (ii) *For  $n \geq 5$ ,  $WB'_n$  is perfect.*

Finite generation of  $WB'_n$  is useful information and exploiting it we have the following.

**Corollary 1.2.** *For any  $n \geq 3$ ,  $WB'_n$  is Hopfian.*

Recall that a group  $G$  is called *Hopfian* if every epimorphism  $G \rightarrow G$  is an isomorphism. In general, being Hopfian is not a subgroup-closed group property.

We also have the following about the homomorphic image of  $WB_n$  into a free group.

**Corollary 1.3.** *For a free group  $F_k$ , the image of any nontrivial homomorphism  $\phi : WB_n \rightarrow F_k$  is infinite cyclic.*

It is evident that for all  $n$ , the commutator subgroups of the flat virtual braid group  $FVB_n$  and the flat welded braid group  $FWB_n$ , are finitely presented as they are finite index subgroups of finitely presented groups. However, we have not seen any presentation of the commutator subgroups of these groups that is noted in the literature. Using Reidemeister-Schreier method, we compute explicit presentations of these commutator subgroups. We note here that more information about structures of these groups may be obtained from the structure of the flat braid groups given in [BBD15].

*Adorability in generalized braid groups.* Motivated by the covering theory of aspherical 3-manifolds, Roushon defined the notion of an adorable group: a group  $G$  is called *adorable* if  $G^i/G^{i+1} = 1$  for some  $i$ , where  $G^i = [G^{i-1}, G^{i-1}]$ , and  $G^0 = G$ . The smallest  $i$  for which the above property holds, is called the *degree of adorability* of  $G$ . For more details on these groups, see [Rou02, Rou04]. Applying Roushon's results with part (ii) of the above theorem, we have the following that generalizes the fact that  $B_n$  is adorable of degree 1 for  $n \geq 5$ .

**Corollary 1.4.** *For  $n \geq 5$ ,  $WB_n$  is adorable of degree 1, and for  $n = 3, 4$ ,  $WB_n$  is not adorable. Thus,  $WB'_n$  is perfect if and only if  $n \geq 5$ .*

It also follows that:

**Corollary 1.5.** *The flat virtual braid group  $FVB_n$  and the flat welded braid group  $FWB_n$  are adorable groups of degree 1 for  $n \geq 5$ . In particular, commutator subgroups of these groups are perfect for  $n \geq 5$ .*

After finishing this article, we have come to know about the recent work of Zaremsky [MZ] that implies the finite presentability of  $WB'_n$  for  $n \geq 4$ , see [MZ, Theorem B]. The finite generation of this group for  $n \geq 3$  is also implicit in this work. However, Zaremsky has not obtained any bounds on the rank, nor he has observed properties like perfectness, Hopfianness etc in his work. Zaremsky has used Morse theory of complex symmetric graphs to obtain his results. It would be interesting to obtain an explicit finite presentation of  $WB'_n$ .

## 2. PROOF OF THEOREM 1.1

The Reidemeister-Schreier method is a standard technique to obtain presentations of subgroups, for example, see [MKS04]. Using this method, presentations of some subgroups of classical braid groups and some of the generalised braid groups have been obtained, for example, see [Man97], [Lř0], [MR]. We shall use it to compute a presentation for  $WB'_n$ . We shall first find out a presentation using Reidemeister-Schreier method and then using Tietze transformations will eliminate redundant generators to obtain a finite generating set.

**2.1. Computing the generators.** Define the map  $\phi$ :

$$1 \rightarrow WB'_n \rightarrow WB_n \xrightarrow{\phi} \mathbb{Z} \times \mathbb{Z}_2 \rightarrow 1$$

where, for  $i = 1, \dots, n-1$ ,  $\phi(\sigma_i) = \overline{\sigma_1}$ ,  $\phi(\rho_i) = \overline{\rho_1}$ ; here  $\overline{\sigma_1}$  and  $\overline{\rho_1}$  are the generators of  $\mathbb{Z}$  and  $\mathbb{Z}_2$  respectively when viewing it in the abelianization of  $WB_n$ . Clearly,  $\phi$  does have a section in the above short exact sequence for  $n \geq 3$ , and  $\ker \phi = WB'_n$ . We will denote  $\phi(a)$ , for  $a \in WB_n$ , simply by  $\overline{a}$ .

Consider a Schreier set of coset representatives:

$$\Lambda = \{\sigma_1^i \rho_1^\epsilon \mid i \in \mathbb{Z}, \epsilon \in \{0, 1\}\}.$$

By [MKS04, Theorem 2.7], the group  $WB'_n$  is generated by the set

$$\{S_{\lambda,a} = (\lambda a)(\overline{\lambda a})^{-1} \mid \lambda \in \Lambda, a \in \{\sigma_i, \rho_i \mid i = 1, 2, \dots, n-1\}\}.$$

Choose  $\lambda = \sigma_1^m \rho_1^\epsilon$  from  $\Lambda$ . Let  $a = \sigma_1$ . Then,  $S_{\lambda, \sigma_1} = \sigma_1^m \rho_1^\epsilon \sigma_1 (\sigma_1^{m+1} \rho_1^\epsilon)^{-1} = \sigma_1^m \rho_1^\epsilon \sigma_1 \rho_1^\epsilon \sigma_1^{-m-1} = \sigma_1^m (\rho_1^\epsilon \sigma_1 \rho_1^\epsilon \sigma_1^{-1}) \sigma_1^{-m}$ . In particular,  $S_{\sigma_1^m, \sigma_1} = 1$ . For  $i > 1$ , let  $a = \sigma_i$ . Then  $S_{\sigma_1^m \rho_1^\epsilon, \sigma_i} = \sigma_1^m \rho_1^\epsilon \sigma_i \rho_1^{-\epsilon} \sigma_1^{-m-1}$ . For  $i > 2$ ,  $\sigma_i$  and  $\rho_1$  commute. Hence,  $S_{\sigma_1^m \rho_1^\epsilon, \sigma_i} = \sigma_i \sigma_1^{-1}$ .

Let  $a = \rho_1$ . Then  $S_{\sigma_1^m \rho_1^\epsilon, \rho_1} = \sigma_1^m \rho_1^{\epsilon+1} (\sigma_1^m \rho_1^{\epsilon+1})^{-1} = 1$ .

For  $a = \rho_i$ ,  $i > 2$ ,  $S_{\sigma_1^m \rho_1^\epsilon, \rho_i} = \sigma_1^m \rho_1^\epsilon \rho_i \rho_1^{-1-\epsilon} \sigma_1^{-m}$ . Note that  $\rho_1$  and  $\rho_i$  commute for  $i > 2$ .

Hence,  $WB'_n$  is generated by the following elements:

$$\begin{aligned} a_m &= S_{\sigma_1^m \rho_1, \sigma_1} = \sigma_1^m (\rho_1 \sigma_1 \rho_1 \sigma_1^{-1}) \sigma_1^{-m}, \\ b_{m, \epsilon} &= S_{\sigma_1^m \rho_1^\epsilon, \sigma_2} = \sigma_1^m (\rho_1^\epsilon \sigma_2 \rho_1^\epsilon \sigma_1^{-1}) \sigma_1^{-m}, \\ c_i &= S_{\sigma_1^m \rho_1^\epsilon, \sigma_i} = \sigma_i \sigma_1^{-1} \text{ (appears only when } n \geq 4), \\ f_{m, \epsilon} &= S_{\sigma_1^m \rho_1^\epsilon, \rho_2} = \sigma_1^m (\rho_1^\epsilon \rho_2 \rho_1^{1-\epsilon}) \sigma_1^{-m}, \\ g_{m, i} &= S_{\sigma_1^m \rho_1^\epsilon, \rho_i} = \sigma_1^m (\rho_i \rho_1) \sigma_1^{-m} \text{ (appears only when } n \geq 4), \end{aligned}$$

where  $m \in \mathbb{Z}$ ,  $\epsilon \in \{0, 1\}$ ,  $3 \leq i \leq n-1$ .

**2.2. Computing the defining relations.** To obtain defining relations of  $WB'_n$ , we define a re-writing process  $\tau$ . By [MKS04, Theorem 2.9], the group  $WB'_n$  is defined by the relations:

$$\tau_{\mu, \lambda} = \tau(\lambda r_\mu \lambda^{-1}), \lambda \in \Lambda,$$

where  $r_\mu$  are the defining relations of  $WB_n$ :

$$\begin{aligned} r_1 &= \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} = 1, \quad |i - j| > 1 \\ r_2 &= \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1 \\ r_3 &= \rho_i^2 = 1 \\ r_4 &= \rho_i \rho_j \rho_i \rho_j = 1, \quad |i - j| > 1 \\ r_5 &= \rho_i \rho_{i+1} \rho_i \rho_{i+1} \rho_i \rho_{i+1} = 1 \\ r_6 &= \sigma_i \rho_j \sigma_i \rho_j^{-1} = 1, \quad |i - j| > 1 \\ r_7 &= \rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i \sigma_{i+1}^{-1} = 1 \\ r_8 &= \rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1 \end{aligned}$$

**Lemma 2.1.**  $WB'_n$  is generated by  $a_k, b_{k,0}, b_{k,1}, c_r, f_{k,0}, f_{k,1} g_{k,r}$  for  $k \in \mathbb{Z}$ ,  $3 \leq r \leq n-1$ , with the following set of defining relations:

$$(2.2.1) \quad b_{k,0} c_l b_{k+1,0}^{-1} c_l^{-1} = 1, \quad l \geq 4;$$

$$(2.2.2) \quad b_{k,1} c_l b_{k+1,1}^{-1} c_l^{-1} = 1, \quad l \geq 4;$$

$$(2.2.3) \quad c_r c_s c_r^{-1} c_s^{-1} = 1, \quad |r - s| > 1;$$

$$(2.2.4) \quad a_k c_r a_{k+1}^{-1} c_r^{-1} = 1;$$

$$(2.2.5) \quad b_{k+1,0} = b_{k,0} b_{k+2,0} ;$$

$$(2.2.6) \quad a_k b_{k+1,1} a_{k+2} = b_{k,1} a_{k+1} b_{k+2,1} ;$$

- (2.2.7)  $b_{k,0}c_3b_{k+2,0} = c_3b_{k+1,0}c_3$  ;
- (2.2.8)  $b_{k,1}c_3b_{k+2,1} = c_3b_{k+1,1}c_3$  ;
- (2.2.9)  $c_rc_{r+1}c_r = c_{r+1}c_rc_{r+1}$  ;
- (2.2.10)  $f_{k,0}f_{k,1} = 1$ ;
- (2.2.11)  $g_{k,r}^2 = 1$ ;
- (2.2.12)  $(f_{k,0}g_{k,l})^2 = 1 = (f_{k,1}g_{k,l})^2, \quad l \geq 4$ ;
- (2.2.13)  $(g_{k,r}g_{k,s})^2 = 1, \quad |r-s| > 1$ ;
- (2.2.14)  $f_{k,1}^3 = 1$ ;
- (2.2.15)  $(f_{k,0}g_{k,3})^3 = 1$ ;
- (2.2.16)  $(g_{k,r}g_{k,r+1})^3 = 1$ ;
- (2.2.17)  $a_kg_{k+1,r}g_{k,r} = 1$ ;
- (2.2.18)  $b_{k,0}g_{k+1,l}b_{k,1}^{-1}g_{k,l} = 1, \quad l \geq 4$ ;
- (2.2.19)  $b_{k,1}g_{k+1,l}b_{k,0}^{-1}g_{k,l} = 1, \quad l \geq 4$ ;
- (2.2.20)  $c_rg_{k+1,s}c_r^{-1}g_{k,s} = 1, \quad |r-s| > 1$ ;
- (2.2.21)  $f_{k,0}c_l f_{k+1,1}c_l^{-1} = 1, \quad l \geq 4$ ;
- (2.2.22)  $f_{k,1}c_l f_{k+1,0}c_l^{-1} = 1, \quad l \geq 4$ ;
- (2.2.23)  $g_{k,r}c_sg_{k+1,r}c_s^{-1} = 1, \quad |r-s| > 1$ ;
- (2.2.24)  $f_{k+1,0}b_{k,0}^{-1}f_{k,1} = 1$ ;
- (2.2.25)  $a_kf_{k+1,1}b_{k,1}^{-1}f_{k,0} = 1$ ;
- (2.2.26)  $b_{k,0}g_{k+1,3}f_{k+1,1}c_3^{-1}f_{k,0}g_{k,3} = 1$ ;
- (2.2.27)  $b_{k,1}g_{k+1,3}f_{k+1,0}c_3^{-1}f_{k,1}g_{k,3} = 1$ ;
- (2.2.28)  $c_rg_{k+1,r+1}g_{k+1,r}c_{r+1}^{-1}g_{k,r}g_{k,r+1} = 1$ ;
- (2.2.29)  $b_{k,0}f_{k+2,0}a_{k+1}^{-1}b_{k,1}^{-1} = 1$ ;
- (2.2.30)  $c_3b_{k+1,0}g_{k+2,3}b_{k+1,1}^{-1}c_3^{-1}f_{k,1} = 1$ ;
- (2.2.31)  $c_{i+1}c_i g_{k+2,i+1}c_i^{-1}c_{i+1}^{-1}g_{k,i} = 1$ .

*Proof.* By re-writing the conjugates of  $r_1$  (by elements of  $\Lambda$ ) we get the relations:

$$(2.2.1), (2.2.2), (2.2.3), (2.2.4)$$

(Note that,  $1 = \tau(r_1) = S_{1,\sigma_i} S_{\sigma_1,\sigma_j} S_{\sigma_1,\sigma_i}^{-1} S_{1,\sigma_j}^{-1}$ . Similarly we compute  $\tau(\rho_1 r_1 \rho_1)$ ,  $\tau(\sigma_1^k r_1 \sigma_1^{-k})$ ,  $\tau(\sigma_1^k \rho_1 r_1 \rho_1 \sigma_1^{-k})$ .)

By re-writing the conjugates of  $r_2$  (by elements of  $\Lambda$ ) we get the relations:

$$(2.2.5), (2.2.6), (2.2.7), (2.2.8), (2.2.9)$$

(Note that,  $1 = \tau(r_2) = S_{1,\sigma_i} S_{\sigma_1,\sigma_{i+1}} S_{\sigma_1^2,\sigma_i} S_{\sigma_1^2,\sigma_{i+1}}^{-1} S_{\sigma_1,\sigma_i}^{-1} S_{1,\sigma_{i+1}}^{-1}$ . Similarly we compute  $\tau(\rho_1 r_2 \rho_1)$ ,  $\tau(\sigma_1^k r_2 \sigma_1^{-k})$ ,  $\tau(\sigma_1^k \rho_1 r_2 \rho_1 \sigma_1^{-k})$ .)

By re-writing the conjugates of  $r_3$  (by elements of  $\Lambda$ ) we get the relations:

$$(2.2.10), (2.2.11)$$

(Note that,  $1 = \tau(r_3) = S_{1,\rho_i} S_{\rho_1,\rho_i}$ . Similarly we compute  $\tau(\rho_1 r_3 \rho_1)$ ,  $\tau(\sigma_1^k r_3 \sigma_1^{-k})$ ,  $\tau(\sigma_1^k \rho_1 r_3 \rho_1 \sigma_1^{-k})$ .)

By re-writing the conjugates of  $r_4$  (by elements of  $\Lambda$ ) we get the relations:

$$(2.2.12), (2.2.13)$$

(Note that,  $1 = \tau(r_4) = S_{1,\rho_i} S_{\rho_1,\rho_j} S_{1,\rho_i} S_{\rho_1,\rho_j}$ . Similarly we compute  $\tau(\rho_1 r_4 \rho_1)$ ,  $\tau(\sigma_1^k r_4 \sigma_1^{-k})$ ,  $\tau(\sigma_1^k \rho_1 r_4 \rho_1 \sigma_1^{-k})$ .)

By re-writing the conjugates of  $r_5$  (by elements of  $\Lambda$ ) we get the relations:

$$(2.2.14), (2.2.15), (2.2.16)$$

(Note that,  $1 = \tau(r_5) = S_{1,\rho_i} S_{\rho_1,\rho_{i+1}} S_{1,\rho_i} S_{\rho_1,\rho_{i+1}} S_{1,\rho_i} S_{\rho_1,\rho_{i+1}}$ . Similarly we compute  $\tau(\rho_1 r_5 \rho_1)$ ,  $\tau(\sigma_1^k r_5 \sigma_1^{-k})$ ,  $\tau(\sigma_1^k \rho_1 r_5 \rho_1 \sigma_1^{-k})$ .)

By re-writing the conjugates of  $r_6$  (by elements of  $\Lambda$ ) we get the relations:

$$(2.2.17), (2.2.18), (2.2.19), (2.2.20), (2.2.21), (2.2.22), (2.2.23)$$

(Note that,  $1 = \tau(r_6) = S_{1,\sigma_i} S_{\sigma_1,\rho_j} S_{\sigma_1\rho_1,\sigma_i} S_{\sigma_1^2,\rho_j}^{-1}$ . Similarly we compute  $\tau(\rho_1 r_6 \rho_1)$ ,  $\tau(\sigma_1^k r_6 \sigma_1^{-k})$ ,  $\tau(\sigma_1^k \rho_1 r_6 \rho_1 \sigma_1^{-k})$ .)

By re-writing the conjugates of  $r_7$  (by elements of  $\Lambda$ ) we get the relations:

$$(2.2.24), (2.2.25), (2.2.26), (2.2.27), (2.2.28)$$

(Note that,  $1 = \tau(r_7) = S_{1,\rho_i} S_{\rho_1,\rho_{i+1}} S_{1,\sigma_i} S_{\sigma_1,\rho_{i+1}} S_{\sigma_1\rho_1,\rho_i} S_{1,\sigma_{i+1}}^{-1}$ . Similarly we compute  $\tau(\rho_1 r_7 \rho_1)$ ,  $\tau(\sigma_1^k r_7 \sigma_1^{-k})$ ,  $\tau(\sigma_1^k \rho_1 r_7 \rho_1 \sigma_1^{-k})$ .)

By re-writing the conjugates of  $r_8$  (by elements of  $\Lambda$ ) we get the relations:

$$(2.2.29), (2.2.30), (2.2.31)$$

(Note that,  $1 = \tau(r_8) = S_{1,\rho_i} S_{\rho_1,\sigma_{i+1}} S_{\sigma_1\rho_1,\sigma_i} S_{\sigma_1^2\rho_1,\rho_{i+1}} S_{\sigma_1,\sigma_i}^{-1} S_{1,\sigma_{i+1}}^{-1}$ . Similarly we compute  $\tau(\rho_1 r_8 \rho_1)$ ,  $\tau(\sigma_1^k r_8 \sigma_1^{-k})$ ,  $\tau(\sigma_1^k \rho_1 r_8 \rho_1 \sigma_1^{-k})$ .)

So, we have a set of defining relations for  $WB'_n$ , namely relations (2.2.1) to (2.2.31) in the generators  $a_k, b_{k,0}, b_{k,1}, c_r, f_{k,0}, f_{k,1}, g_{k,r}$  for  $k \in \mathbb{Z}$  and  $3 \leq r \leq n-1$ . Hence, Lemma 2.1 is proved.  $\square$

Now, we will eliminate some of the generators and relations through Tietze transformations in order to have a finite set of generators for  $WB'_n$ .

### 2.3. Proof of Theorem 1.1.

*Proof.* From (2.2.25) we have  $b_{k,1} = f_{k,0} a_k f_{k+1,1}$ . We remove  $b_{k,1}$  by replacing this value in all other relations. After this replacement, (2.2.29) becomes:

$$f_{k+1,1}^{-1} a_k^{-1} f_{k,0}^{-1} b_{k,0} f_{k+2,0} = a_{k+1}$$

From this relation, we can express  $a_k$  in terms of  $a_0, b_{k,0}, f_{k,0}, f_{k,1}$ . We replace this value of  $a_k$  in all other relations and remove  $a_k$  for all  $k \neq 0$ .

Using (2.2.24) we get  $f_{k,0} = f_{0,0} b_{0,0} b_{1,0} \dots b_{k-1,0}$  for  $k \geq 1$  and we get for  $k \leq -1$

$$f_{k,0} = f_{0,0} b_{-1,0}^{-1} b_{-2,0}^{-1} \dots b_{k,0}^{-1}.$$

Also we have  $f_{k,1} = f_{k,0}^{-1}$ . We replace these values of  $f_{k,0}$  and  $f_{k,1}$  in terms of  $a_k, b_{k,0}, b_{k,1}, f_{0,0}$  in all the other relations and remove all  $f_{k,1}$  and all  $f_{k,0}$  except  $f_{0,0}$ .

Then, using (2.2.5) we replace all  $b_{k,0}$  in terms of  $b_{0,0}, b_{1,0}$ .

Lastly, if  $n \geq 4$ , using (2.2.17) we can remove  $g_{k,r}$  for  $k \neq 0$ .

Hence, we get a presentation of  $WB'_n$  with  $4 + 2(n-3)$  generators  $a_0, b_{0,0}, b_{1,0}, f_{0,0}, c_r, g_{0,r}$ ,  $3 \leq r \leq n-1$ , and infinitely many defining relations. This proves finite generation of  $WB'_n$  for all  $n$ .

We undergo with an alternative elimination process as follows. Note that for  $n \geq 7$ , for every generator  $g_{k,r}$ , there is at least one  $g_{k,s}$ ,  $|r-s| > 1$ . This will help us to improve the number of generators of  $WB'_n$  for  $n \geq 7$  in the new presentation.

**2.3.1. Alternative Elimination.** Note that,  $b_{k,0} = f_{k,1} f_{k+1,0}$  and  $b_{k,1} = f_{k,0} a_k f_{k+1,1}$ . Also note that,  $a_k = g_{k,r} g_{k+1,r}$ . At first, we replace  $b_{k,0}$  and  $b_{k,1}$  by  $f_{k,1} f_{k+1,0}$  and  $f_{k,0} a_k f_{k+1,1}$  in all the above relations and remove these generators from the set of generators. Next, we replace  $a_k$  by  $g_{k,r} g_{k+1,r}$  (for every  $3 \leq r \leq n-1$ ) in the current set of relations and remove these generators from the current set of generators, and we have the new set of defining relations in the generators  $f_{k,0}, f_{k,1}, g_{k,r}, c_r$ , for all  $k \in \mathbb{Z}$  and  $3 \leq r \leq n-1$ :

$$(2.3.1) \quad g_{k,r} g_{k+1,r} = g_{k,s} g_{k+1,s} \quad \forall r, s;$$

$$(2.3.2) \quad f_{k+1,1} f_{k+2,0} = f_{k,1} f_{k+1,0} f_{k+2,1} f_{k+3,0};$$

$$(2.3.3) \quad g_{k,r} g_{k+1,r} f_{k+1,0} g_{k+1,r} g_{k+2,r} f_{k+2,1} g_{k+2,r} g_{k+3,r} = f_{k,0} g_{k,r} g_{k+1,r} f_{k+1,1} g_{k+1,r} g_{k+2,r} f_{k+2,0} g_{k+2,r} g_{k+3,r} f_{k+3,1};$$

$$(2.3.4) \quad f_{k,1} f_{k+1,0} c_3 f_{k+2,1} f_{k+3,0} = c_3 f_{k+1,1} f_{k+2,0} c_3;$$

$$(2.3.5) \quad f_{k,0} g_{k,r} g_{k+1,r} f_{k+1,1} c_3 f_{k+2,0} g_{k+2,r} g_{k+3,r} f_{k+3,1} = c_3 f_{k+1,0} g_{k+1,r} g_{k+2,r} f_{k+2,1} c_3;$$

$$(2.3.6) \quad c_r c_{r+1} c_r = c_{r+1} c_r c_{r+1} ;$$

$$(2.3.7) \quad f_{k,1}^3 = 1; \quad f_{k,0} f_{k,1} = 1;$$

$$(2.3.8) \quad g_{k,r}^2 = 1; \quad (f_{k,0} g_{k,3})^3 = 1; \quad (g_{k,r} g_{k,r+1})^3 = 1;$$

$$(2.3.9) \quad (f_{k,0} g_{k,l})^2 = 1, \quad l \geq 4;$$

$$(2.3.10) \quad (g_{k,r} g_{k,s})^2 = 1, \quad |r - s| > 1;$$

$$(2.3.11) \quad f_{k,1} f_{k+1,0} c_l f_{k+2,0}^{-1} f_{k+1,1}^{-1} c_l^{-1} = 1, \quad l \geq 4;$$

$$(2.3.12) \quad f_{k,0} g_{k,r} g_{k+1,r} f_{k+1,1} c_l f_{k+2,1}^{-1} g_{k+2,r}^{-1} g_{k+1,r}^{-1} f_{k+1,0}^{-1} c_l^{-1} = 1, \quad l \geq 4;$$

$$(2.3.13) \quad c_r c_s c_r^{-1} c_s^{-1} = 1, \quad |r - s| > 1;$$

$$(2.3.14) \quad g_{k,r} g_{k+1,r} c_j g_{k+2,r}^{-1} g_{k+1,r}^{-1} c_j^{-1} = 1;$$

$$(2.3.15) \quad f_{k,1} f_{k+1,0} g_{k+1,l} f_{k+1,1}^{-1} g_{k+1,r} g_{k,r} f_{k,0}^{-1} g_{k,l} = 1, \quad l \geq 4;$$

$$(2.3.16) \quad f_{k,0} c_l f_{k+1,1} c_l^{-1} = 1, \quad l \geq 4;$$

$$(2.3.17) \quad g_{k,r} c_s g_{k+1,r} c_s^{-1} = 1, \quad |r - s| > 1;$$

$$(2.3.18) \quad f_{k,1} f_{k+1,0} g_{k+1,3} f_{k+1,1} c_3^{-1} f_{k,0} g_{k,3} = 1;$$

$$(2.3.19) \quad f_{k,0} g_{k,r} g_{k+1,r} f_{k+1,1} g_{k+1,3} f_{k+1,0} c_3^{-1} f_{k,1} g_{k,3} = 1;$$

$$(2.3.20) \quad c_r g_{k+1,r+1} g_{k+1,r} c_{r+1}^{-1} g_{k,r} g_{k,r+1} = 1;$$

$$(2.3.21) \quad f_{k,1} f_{k+1,0} f_{k+2,0} g_{k+2,r} g_{k+1,r} f_{k+1,1}^{-1} g_{k+1,r} g_{k,r} f_{k,0}^{-1} = 1;$$

$$(2.3.22) \quad c_3 f_{k+1,1} f_{k+2,0} g_{k+2,3} f_{k+2,1}^{-1} g_{k+2,r} g_{k+1,r} f_{k+1,0}^{-1} c_3^{-1} f_{k,1} = 1;$$

$$(2.3.23) \quad c_{i+1} c_i g_{k+2,i+1} c_i^{-1} c_{i+1}^{-1} g_{k,i} = 1.$$

Now the number of generators can be improved for  $n \geq 7$ .

**Lemma 2.2.** *For  $n \geq 7$ ,  $WB'_n$  is generated by  $2(n-3) + 1$  elements  $f_{0,0}, g_{0,r}, c_r$  for  $3 \leq r \leq n-1$ .*

*Proof.* We assume  $k \geq 0$ .  $k < 0$  case is similar. For  $n \geq 7$ , note that  $g_{k+1,r} = c_s^{-1} g_{k,r} c_s$ ,  $|r - s| > 1$ . Hence, we have  $g_{k,r} = c_s^{-k} g_{0,r} c_s^k$ . Also, note that,  $f_{0,1} = f_{0,0}^{-1}$ . We replace  $f_{k,0}, f_{k,1}, g_{k,r}$  by  $c_l^{-k} f_{0,0} c_l^k$ ,  $c_l^{-k} f_{0,0}^{-1} c_l^k$ ,  $c_s^{-k} g_{0,r} c_s^k$  in the current set of relations and remove  $f_{0,1}, f_{k,0}, f_{k,1}, g_{k,r}$ , for all  $k \neq 0$ , from the set of generators and we get the new set of defining relations in the generators  $f_{0,0}, g_{0,r}, c_r$  for  $3 \leq r \leq n-1$ . This proves the lemma.  $\square$



*Perfectness of the commutator subgroup.* For  $n \geq 5$ , we abelianise the above presentation of  $WB'_n$  by adding the extra relations  $xy = yx$  for all  $x, y$  in the generating set. After abelianising, we see from (2.3.6) that,  $c_r = c_{r+1}$  for all  $r$ , and hence from (2.3.16),  $f_{k+1,0} = f_{k,0}$ . From (2.3.4), we see that  $c_3 = 1$ , hence  $c_r = 1$  for all  $r$ . From (2.3.9), it follows that in the abelianization,  $f_{k,0}^2 = 1$ , hence using  $f_{k,0}^3 = 1$  we see that  $f_{k,0} = 1$  for all  $k$ . From (2.3.8) that  $g_{k,r} = 1$  for all  $k$ . Thus the abelianization of  $WB'_n$  is the identity group. This shows that for  $n \geq 5$ ,  $WB'_n$  is perfect.  $\square$

#### 2.4. Proof of the Corollaries.

2.4.1. *Proof of Corollary 1.2.* Using [Dam16, Corollary 4.3], it follows that  $WB_n$  is a subgroup of  $\text{Aut}(F_n)$ , and hence so also  $WB'_n$ . Since  $F_n$  is residually finite, using a result of Magnus [Mag69] it follows that  $\text{Aut}(F_n)$  is also residually finite. Hence  $WB'_n$  as a subgroup of  $\text{Aut}(F_n)$  is also residually finite. It is well-known that a finitely generated residually finite group is Hopfian. Thus  $WB'_n$  is Hopfian for all  $n$ .

2.4.2. *Proof of Corollary 1.3.* Suppose  $\phi : WB_n \rightarrow F_k$  be a nontrivial homomorphism. By Theorem 1.1,  $WB'_n$  is finitely generated. Hence,  $\phi(WB'_n) = \phi(WB_n)'$  is finitely generated. But,  $\phi(WB_n)$  is free group of finite rank. Hence,  $\phi(WB_n)'$  is finitely generated only if rank of  $\phi(WB_n)$  is at most 1. This proves Corollary 1.2.

2.4.3. *Proof of Corollary 1.4.* It follows from [Bar03] that there is a non-trivial homomorphism from the pure welded braid group,  $PWB_n$ , onto a free group. Hence  $PWB_n$  is not adorable. For  $k = 3, 4$ ,  $WB_k/PWB_k$  is a finite solvable group. Hence, by [Rou04, Proposition 1.7], for  $k = 3, 4$ ,  $WB_k$  is not adorable, in particular,  $WB'_k$  is not perfect for  $k = 3, 4$ . This proves the corollary.

2.4.4. *Proof of Corollary 1.5.* It is easy to see that if  $f : G \rightarrow H$  be a surjective homomorphism with  $G$  adorable, then  $H$  is also adorable and  $\text{doa}(H) \leq \text{doa}(G)$ , where  $\text{doa}(G)$  denotes degree of adorability, see [Rou04, Lemma 1.1]. It follows from [BB09, Proposition 8] that the commutator subgroup  $VB'_n$  of the virtual braid group  $VB_n$  is perfect for  $n \geq 5$ . Thus, for  $n \geq 5$ ,  $VB_n$  is adorable of degree 1. The flat braid groups being quotients of these groups, are also adorable with degree  $\leq 1$ . Since the flat groups are not perfect, they are adorable of degree 1. In particular, the commutator subgroups of these groups are perfect.

These arguments also imply that  $WB'_n$  is perfect for  $n \geq 5$  without using the presentation of  $WB'_n$ .

### 3. COMMUTATORS OF FLAT BRAID GROUPS

**Theorem 3.1.**  *$FVB'_n$  and  $FWB'_n$  are finitely presented groups for all  $n$ .*

*$FVB'_n$  has the following finite presentation:*

*Set of Generators:*

$$c_1, a_2, b_2, c_2, f_2, a_i, b_i, \quad i = 3, \dots, n-1$$

*Set of Defining Relations:*

$$\begin{aligned} a_2^3 &= b_2^3 = c_2^3 = f_2^3 = 1; \\ a_i^2 &= b_i^2 = (b_i c_1)^2 = 1, \quad i = 3, \dots, n-1; \\ b_2^{-1} f_2 a_2^{-1} &= 1; \end{aligned}$$

$$\begin{aligned}
& b_2 c_1 f_2^{-1} c_2^{-1} = 1; \\
& (a_2 a_i)^2 = (b_2 b_i)^2 = (c_2 a_i)^2 = (f_2 b_i c_1)^2 = 1, i \geq 4; \\
& (a_2 a_3)^3 = (b_2 b_3)^3 = (c_2 a_3)^3 = (f_2 b_3 c_1)^3 = 1; \\
& a_2 b_i c_1 = b_i c_2, i \geq 4; \\
& a_i f_2 = b_2 a_i, i \geq 4; \\
& b_2 b_3 a_2 b_3 c_1 f_2^{-1} a_3 = 1; \\
& b_2^{-1} b_3 c_2 b_3 c_1 f_2 a_3 = 1; \\
& (a_i a_j)^2 = (b_i b_j)^2 = 1, i, j \geq 3, |i - j| > 1; \\
& (a_i a_{i+1})^3 = (b_i b_{i+1})^3 = 1, i \geq 3; \\
& b_j^{-1} a_i^{-1} b_j a_i = c_1, i, j \geq 3, |i - j| > 1; \\
& b_i b_{i+1} a_i = a_{i+1} b_i b_{i+1}, i \geq 3.
\end{aligned}$$

The group  $FWB'_n$  is generated by the same set of generators as above and has set of defining relations as the same set of relations above along with the following relations:

$$\begin{aligned}
& a_2^{-1} c_2 c_1^{-1} b_2^{-1} = 1; \\
& a_2 c_2^{-1} c_1 f_2^{-1} = 1; \\
& a_2 a_3 b_2 a_3 c_2^{-1} b_3 = 1; \\
& a_2^{-1} a_3 f_2 a_3 c_2 b_3 c_1 = 1; \\
& a_2 b_i c_1 = b_i c_2, i \geq 4.
\end{aligned}$$

In particular, for  $n = 3$ , we have the following presentations:

$$\begin{aligned}
FVB'_3 &= \langle a, b, x, y \mid a^3 = b^3 = (ab)^3 = (xy)^3 = 1, y^{-1} = axb \rangle \\
FWB'_3 &= \langle a, b, c, x \mid a^3 = b^3 = c^3 = 1, abc = 1, axc = bax = xcb \rangle
\end{aligned}$$

**3.1. Computing the generators.** Let  $G = FVB_n$  or  $FWB_n$ . Define the map  $\phi$ :

$$1 \rightarrow G' \rightarrow G \xrightarrow{\phi} \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow 1$$

where, for  $i = 1, \dots, n-1$ ,  $\phi(\sigma_i) = \overline{\sigma_1}$ ,  $\phi(\rho_i) = \overline{\rho_1}$ ; here  $\overline{\sigma_1}$  and  $\overline{\rho_1}$  are the generators of the two copies of  $\mathbb{Z}_2$  when viewing it in the abelianization of  $G$ . Clearly,  $\phi$  does have a section in the above short exact sequence and  $\ker \phi = G'$ . We will denote  $\phi(a)$ , for  $a \in G$ , simply by  $\overline{a}$ .

Consider a Schreier set of coset representatives:

$$\Lambda = \{1, \sigma_1, \rho_1, \sigma_1 \rho_1\}.$$

By [MKS04, Theorem 2.7], the group  $G'$  is generated by the set:

$$\{S_{\lambda, a} = (\lambda a)(\overline{\lambda a})^{-1} \mid \lambda \in \Lambda, a \in \{\sigma_i, \rho_i \mid i = 1, 2, \dots, n-1\}\}$$

We denote the generators as follows:

(1)  $\lambda = 1$ :

$$\begin{aligned}
S_{1, \sigma_i} &= \sigma_i \sigma_1 = a_i \\
S_{1, \rho_i} &= \rho_i \rho_1 = b_i
\end{aligned}$$

(2)  $\lambda = \rho_1$ :

$$\begin{aligned} S_{\rho_1, \sigma_i} &= \rho_1 \sigma_i \rho_1 \sigma_1 = c_i \\ S_{\rho_1, \rho_i} &= \rho_1 \rho_i = d_i \end{aligned}$$

(3)  $\lambda = \sigma_1$ :

$$\begin{aligned} S_{\sigma_1, \sigma_i} &= \sigma_1 \sigma_i = e_i \\ S_{\sigma_1, \rho_i} &= \sigma_1 \rho_i \rho_1 \sigma_1 = f_i \end{aligned}$$

(4)  $\lambda = \sigma_1 \rho_1$ :

$$\begin{aligned} S_{\sigma_1 \rho_1, \sigma_i} &= \sigma_1 \rho_1 \sigma_i \rho_1 = g_i \\ S_{\sigma_1 \rho_1, \rho_i} &= \sigma_1 \rho_1 \rho_i \sigma_1 = h_i \end{aligned}$$

In the rest of this section, we shall derive the presentations using Reidemeister-Schreier method.

**3.2. Computing the defining relations.** To obtain defining relations of  $G'$ , we define a re-writing process  $\tau$ . By [MKS04, Theorem 2.9], the group  $G'$  is defined by the relations:

$$\tau_{\mu, \lambda} = \tau(\lambda r_{\mu} \lambda^{-1}), \lambda \in \Lambda,$$

where  $r_{\mu}$  are the defining relations of  $G$ .

The defining relations for  $FVB_n$  are:

$$\begin{aligned} r_1 &= \sigma_i \sigma_j \sigma_i \sigma_j = 1, |i - j| > 1 \\ r_2 &= \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1} = 1 \\ r_3 &= \sigma_i^2 = 1 \\ r_4 &= \rho_i^2 = 1 \\ r_5 &= \rho_i \rho_j \rho_i \rho_j = 1, |i - j| > 1 \\ r_6 &= \rho_i \rho_{i+1} \rho_i \rho_{i+1} \rho_i \rho_{i+1} = 1 \\ r_7 &= \sigma_i \rho_j \sigma_i \rho_j = 1, |i - j| > 1 \\ r_8 &= \rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i \sigma_{i+1} = 1 \end{aligned}$$

There is one extra defining relation for  $FWB_n$ :

$$r_9 = \rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i \sigma_{i+1} = 1$$

Applying the re-writing process on the above relations we get:

$$\begin{aligned} 1 &= r_1 = S_{1, \sigma_i} S_{\sigma_1, \sigma_j} S_{1, \sigma_i} S_{\sigma_1, \sigma_j} = (a_i e_j)^2 ; \\ 1 &= r_2 = S_{1, \sigma_i} S_{\sigma_1, \sigma_{i+1}} S_{1, \sigma_i} S_{\sigma_1, \sigma_{i+1}} S_{1, \sigma_i} S_{\sigma_1, \sigma_{i+1}} = (a_i e_{i+1})^3 ; \\ 1 &= r_3 = S_{1, \sigma_i} S_{\sigma_1, \sigma_i} = a_i e_i ; \\ 1 &= r_4 = S_{1, \rho_i} S_{\rho_1, \rho_i} = b_i d_i ; \\ 1 &= r_5 = S_{1, \rho_i} S_{\rho_1, \rho_j} S_{1, \rho_i} S_{\rho_1, \rho_j} = (b_i d_j)^2 ; \\ 1 &= r_6 = S_{1, \rho_i} S_{\rho_1, \rho_{i+1}} S_{1, \rho_i} S_{\rho_1, \rho_{i+1}} S_{1, \rho_i} S_{\rho_1, \rho_{i+1}} = (b_i d_{i+1})^3 ; \\ 1 &= r_7 = S_{1, \sigma_i} S_{\sigma_1, \rho_j} S_{\sigma_1 \rho_1, \sigma_i} S_{\rho_1, \rho_j} = a_i f_j g_i d_j ; \\ 1 &= r_8 = S_{1, \rho_i} S_{\rho_1, \rho_{i+1}} S_{1, \sigma_i} S_{\sigma_1, \rho_{i+1}} S_{\sigma_1 \rho_1, \rho_i} S_{\sigma_1, \sigma_{i+1}} = b_i d_{i+1} a_i f_{i+1} h_i e_{i+1} ; \\ 1 &= r_9 = S_{1, \rho_i} S_{\rho_1, \sigma_{i+1}} S_{\sigma_1 \rho_1, \sigma_i} S_{\rho_1, \rho_{i+1}} S_{1, \sigma_i} S_{\sigma_1, \sigma_{i+1}} = b_i c_{i+1} g_i d_{i+1} a_i e_{i+1} . \end{aligned}$$

Conjugating the defining relations by  $\sigma_1$  and re-writing them we get:

$$\begin{aligned}
1 &= \sigma_1 r_1 \sigma_1 = (e_i a_j)^2 ; \\
1 &= \sigma_1 r_2 \sigma_1 = (e_i a_{i+1})^3 ; \\
1 &= \sigma_1 r_3 \sigma_1 = e_i a_i ; \\
1 &= \sigma_1 r_4 \sigma_1 = f_i h_i ; \\
1 &= \sigma_1 r_5 \sigma_1 = (f_i h_j)^2 ; \\
1 &= \sigma_1 r_6 \sigma_1 = (f_i h_{i+1})^3 ; \\
1 &= \sigma_1 r_7 \sigma_1 = e_i b_j c_i h_j ; \\
1 &= \sigma_1 r_8 \sigma_1 = f_i h_{i+1} e_i b_{i+1} d_i a_{i+1} ; \\
1 &= \sigma_1 r_9 \sigma_1 = f_i g_{i+1} c_i h_{i+1} e_i a_{i+1} .
\end{aligned}$$

Conjugating the defining relations by  $\rho_1$  and re-writing them we get:

$$\begin{aligned}
1 &= \rho_1 r_1 \rho_1 = (c_i g_j)^2 ; \\
1 &= \rho_1 r_2 \rho_1 = (c_i g_{i+1})^3 ; \\
1 &= \rho_1 r_3 \rho_1 = c_i g_i ; \\
1 &= \rho_1 r_4 \rho_1 = d_i b_i ; \\
1 &= \rho_1 r_5 \rho_1 = (d_i b_j)^2 ; \\
1 &= \rho_1 r_6 \rho_1 = (d_i b_{i+1})^3 ; \\
1 &= \rho_1 r_7 \rho_1 = c_i h_j e_i b_j ; \\
1 &= \rho_1 r_8 \rho_1 = d_i b_{i+1} c_i h_{i+1} f_i g_{i+1} ; \\
1 &= \rho_1 r_9 \rho_1 = d_i a_{i+1} e_i b_{i+1} c_i g_{i+1} .
\end{aligned}$$

Conjugating the defining relations by  $\sigma_1 \rho_1$  and re-writing them we get:

$$\begin{aligned}
1 &= \sigma_1 \rho_1 r_1 \rho_1 \sigma_1 = (g_i c_j)^2 ; \\
1 &= \sigma_1 \rho_1 r_2 \rho_1 \sigma_1 = (g_i c_{i+1})^3 ; \\
1 &= \sigma_1 \rho_1 r_3 \rho_1 \sigma_1 = g_i c_i ; \\
1 &= \sigma_1 \rho_1 r_4 \rho_1 \sigma_1 = h_i f_i ; \\
1 &= \sigma_1 \rho_1 r_5 \rho_1 \sigma_1 = (h_i f_j)^2 ; \\
1 &= \sigma_1 \rho_1 r_6 \rho_1 \sigma_1 = (h_i f_{i+1})^3 ; \\
1 &= \sigma_1 \rho_1 r_7 \rho_1 \sigma_1 = g_i d_j a_i f_j ; \\
1 &= \sigma_1 \rho_1 r_8 \rho_1 \sigma_1 = h_i f_{i+1} g_i d_{i+1} b_i c_{i+1} ; \\
1 &= \sigma_1 \rho_1 r_9 \rho_1 \sigma_1 = h_i e_{i+1} a_i f_{i+1} g_i c_{i+1} .
\end{aligned}$$

Replacing  $e_i, d_i, g_i, h_i$  by  $a_i^{-1}, b_i^{-1}, c_i^{-1}, f_i^{-1}$  respectively, we get the defining relations for  $FVB'_n$  in terms of the generators  $a_i, b_i, c_i, f_i$  as follows:

$$(a_i a_j^{-1})^2 = 1, \quad |i - j| > 1;$$

$$\begin{aligned}
(b_i b_j^{-1})^2 &= 1, \quad |i - j| > 1; \\
(c_i c_j^{-1})^2 &= 1, \quad |i - j| > 1; \\
(f_i f_j^{-1})^2 &= 1, \quad |i - j| > 1;
\end{aligned}$$

$$\begin{aligned}
(a_i a_{i+1}^{-1})^3 &= 1; \\
(b_i b_{i+1}^{-1})^3 &= 1; \\
(c_i c_{i+1}^{-1})^3 &= 1; \\
(f_i f_{i+1}^{-1})^3 &= 1;
\end{aligned}$$

$$\begin{aligned}
a_i f_j &= b_j c_i, \quad |i - j| > 1; \\
a_1 &= b_1 = f_1 = 1;
\end{aligned}$$

$$\begin{aligned}
b_i b_{i+1}^{-1} a_i f_{i+1} f_i^{-1} a_{i+1}^{-1} &= 1; \\
b_i^{-1} b_{i+1} c_i f_{i+1}^{-1} f_i c_{i+1}^{-1} &= 1.
\end{aligned}$$

For  $FWB'_n$  we have two extra defining relations:

$$\begin{aligned}
a_i a_{i+1}^{-1} b_i c_{i+1} c_i^{-1} b_{i+1}^{-1} &= 1; \\
a_i^{-1} a_{i+1} f_i c_{i+1}^{-1} c_i f_{i+1}^{-1} &= 1.
\end{aligned}$$

Observe that, putting  $i = 1$  in the first 8 relations, we get the following:

$$\begin{aligned}
a_j^2 &= b_j^2 = c_j^2 = f_j^2 = 1, \quad j = 3, \dots, n-1, \\
a_2^3 &= b_2^3 = c_2^3 = f_2^3 = 1.
\end{aligned}$$

Note that, if  $|i - j| > 1$ , we have  $a_i f_j = b_j c_i$ . Putting  $j = 1$ , we get  $c_i = a_i$  for  $i = 3, \dots, n-1$ . And putting  $i = 1$ , we get  $f_j = b_j c_1$  for  $j = 3, \dots, n-1$ .

We replace  $c_i$  by  $a_i$  and  $f_i$  by  $b_i c_1$  for  $i = 3, \dots, n-1$ .

Putting  $i = 1$  in the next 2 relations, we have:

$$\begin{aligned}
b_2^{-1} f_2 a_2^{-1} &= 1; \\
b_2 c_1 f_2^{-1} c_2^{-1} &= 1.
\end{aligned}$$

Putting  $i = 1$  in the two extra relations of  $FWB'_n$ , we have:

$$\begin{aligned}
a_2^{-1} c_2 c_1^{-1} b_2^{-1} &= 1; \\
a_2 c_2^{-1} c_1 f_2^{-1} &= 1.
\end{aligned}$$

Similarly, considering the cases  $i = 2, j \geq 4$ , and,  $i \geq 4, j = 2$ , in the above relations we get the following set of relations for  $FVB'_n$ :

$$(a_2a_i)^2 = (b_2b_i)^2 = (c_2a_i)^2 = (f_2b_ic_1)^2 = 1, \quad i \geq 4;$$

$$(a_2a_3)^3 = (b_2b_3)^3 = (c_2a_3)^3 = (f_2b_3c_1)^3 = 1;$$

$$a_2b_ic_1 = b_ic_2, \quad i \geq 4;$$

$$a_if_2 = b_2a_i, \quad i \geq 4;$$

$$b_2b_3a_2b_3c_1f_2^{-1}a_3 = 1;$$

$$b_2^{-1}b_3c_2b_3c_1f_2a_3 = 1.$$

And, the two extra relations for  $FWB'_n$ :

$$a_2a_3b_2a_3c_2^{-1}b_3 = 1 ;$$

$$a_2^{-1}a_3f_2a_3c_2b_3c_1 = 1.$$

Lastly, we consider the case  $i, j \geq 3$ :

$$(a_ia_j)^2 = (b_ib_j)^2 = 1, \quad i, j \geq 3, \quad |i - j| > 1;$$

$$(a_ia_{i+1})^3 = (b_ib_{i+1})^3 = 1, \quad i \geq 3;$$

$$b_j^{-1}a_i^{-1}b_ja_i = c_1, \quad i, j \geq 3, \quad |i - j| > 1;$$

$$b_ib_{i+1}a_i = a_{i+1}b_ib_{i+1}, \quad i \geq 3.$$

And, the two extra relations for  $FWB'_n$ :

$$a_ia_{i+1}b_i = b_{i+1}a_ia_{i+1}, \quad i \geq 3;$$

$$c_1a_ia_{i+1}b_ic_1 = b_{i+1}a_ia_{i+1}, \quad i \geq 3.$$

This completes the proof of Theorem 3.1.

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## REFERENCES

- [Bar03] V. G. Bardakov. The structure of a group of conjugating automorphisms. *Algebra Logika*, 42(5):515–541, 636, 2003.
- [BB09] Valerij G. Bardakov and Paolo Bellingeri. Combinatorial properties of virtual braids. *Topology Appl.*, 156(6):1071–1082, 2009.
- [BBD15] Valeriy G. Bardakov, Paolo Bellingeri, and Celeste Damiani. Unrestricted virtual braids, fused links and other quotients of virtual braid groups. *J. Knot Theory Ramifications*, 24(12):1550063, 23, 2015.
- [BWC07] John C. Baez, Derek K. Wise, and Alissa S. Crans. Exotic statistics for strings in 4D  $BF$  theory. *Adv. Theor. Math. Phys.*, 11(5):707–749, 2007.
- [Col89] Donald J. Collins. Cohomological dimension and symmetric automorphisms of a free group. *Comment. Math. Helv.*, 64(1):44–61, 1989.
- [Dam16] Celeste Damiani. A journey through loop braid groups. *Expositiones Mathematicae*, pages –, 2016.
- [FRR97] Roger Fenn, Richárd Rimányi, and Colin Rourke. The braid-permutation group. *Topology*, 36(1):123–135, 1997.
- [GL69] E. A. Gorin and V. Ja. Lin. Algebraic equations with continuous coefficients, and certain questions of the algebraic theory of braids. *Mat. Sb. (N.S.)*, 78 (120):579–610, 1969.
- [Kam07] Seiichi Kamada. Braid presentation of virtual knots and welded knots. *Osaka J. Math.*, 44(2):441–458, 2007.
- [Kau99] Louis H. Kauffman. Virtual knot theory. *European J. Combin.*, 20(7):663–690, 1999.
- [KL04] Louis H. Kauffman and Sofia Lambropoulou. Virtual braids. *Fund. Math.*, 184:159–186, 2004.
- [L10] Michael Lönne. Presentations of subgroups of the braid group generated by powers of band generators. *Topology Appl.*, 157(7):1127–1135, 2010.
- [Mag69] W. Magnus. Residually finite groups. *Bull. Amer. Math. Soc.*, 75:305–316, 1969.
- [Man97] Sandro Manfredini. Some subgroups of Artin’s braid group. *Topology Appl.*, 78(1-2):123–142, 1997. Special issue on braid groups and related topics (Jerusalem, 1995).
- [MKS04] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory*. Dover Publications, Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.
- [MR] Jamie Mulholland and Dale Rolfsen. Local indicability and commutator subgroups of artin groups. *ArXiv*, arXiv:math/0606116.
- [Ore12] S. Yu. Orevkov. On the commutants of Artin groups. *Dokl. Akad. Nauk*, 442(6):740–742, 2012.
- [Rou02] S. K. Roushon. Topology of 3-manifolds and a class of groups. *Arxiv*, arXiv:math/0209121, 2002.
- [Rou04] S. K. Roushon. Topology of 3-manifolds and a class of groups. II. *Bol. Soc. Mat. Mexicana (3)*, 10(Special Issue):467–485, 2004.
- [Sav93] A. G. Savushkina. On the commutator subgroup of the braid group. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (6):11–14, 118 (1994), 1993.
- [MZ] M. C. B. Zaremsky, Symmetric automorphisms of free groups, BNSR-invariants, and finiteness properties. *ArXiv*, arxiv: 1607.03043, 2016.
- [Zin75] V. M. Zinde. Commutants of Artin groups. *Uspehi Mat. Nauk*, 30(5(185)):207–208, 1975.

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